GENERALISED GROUPS

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A translation of:
Обобшённые Группы, *V. V. Wagner (В. В. Вагнер),*
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Let S be a semigroup[1] in which idempotents commute. Then the set I of all idempotents is a commutative subsemigroup. Notice that for arbitrary elements $s, \overline{s} \in S$, the implication[2] $(s\overline{s}s = s) \Rightarrow \{(s\overline{s}s\overline{s} = s\overline{s}) \wedge (\overline{s}s\overline{s}s = \overline{s}s)\}$ holds, so we obtain

$$(s\overline{s}s = s) \Rightarrow \{(s\overline{s} \in I) \wedge (\overline{s}s \in I)\}. \tag{1}$$

Theorem 1. *For every element s, there exists a* unique *element $\overline{s}$ which satisfies the conditions*

$$s\overline{s}s = s; \quad \overline{s}s\overline{s} = s. \tag{2}$$

Proof. Let $\overline{s}$ and $\overline{s}^*$ both satisfy (2) for a given s. By (1), $s\overline{s}$, $s\overline{s}^*$, $\overline{s}s$, $\overline{s}^*s$ are all idempotent and commute with eachother. From the equations $s\overline{s}s\overline{s}^* = s\overline{s}^*s\overline{s}$ and $\overline{s}s\overline{s}^*s = \overline{s}^*s\overline{s}s$, we obtain $s\overline{s}^* = s\overline{s}$ and $\overline{s}s = \overline{s}^*s$, whence, multiplying the first equation on the left by $\overline{s}^*$ and the second on the right by $\overline{s}$, we have $\overline{s}^* = \overline{s}^*s\overline{s} = \overline{s}$. □

An element s for which there exists an $\overline{s}$ satisfying (2) is said to be *generally invertible*; the element $\overline{s}$ is called a *generalised inverse* for s, and is denoted by s^{-1}.

Using this notation,

$$(s^{-1})^{-1} = s; \quad ss^{-1}s = s; \quad ss^{-1} \in I. \tag{3}$$

It is clear that every idempotent i has a generalised inverse and that $i^{-1} = i$.

Theorem 2. *If s_1 and s_2 are generally invertible, then so is s_1s_2. Moreover*

$$(s_1s_2)^{-1} = s_2^{-1}s_1^{-1}. \tag{4}$$

Proof. By (3),

$$s_1s_2s_2^{-1}s_1^{-1}s_1s_2 = s_1s_1^{-1}s_1s_2s_2^{-1}s_2 = s_1s_2,$$
$$s_2^{-1}s_1^{-1}s_1s_2s_2^{-1}s_1^{-1} = s_2^{-1}s_2s_2^{-1}s_1^{-1}s_1s_1^{-1} = s_2^{-1}s_1^{-1}.$$

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[1]By *semigroup*, we mean a set with an associative binary operation.

[2]*In this paper, Wagner used the symbol · for conjunction of statements. I have adopted the more modern ∧, which was used by Wagner in a later paper* [7].

It therefore follows that $\overline{s} = s_2^{-1}s_1^{-1}$ satisfies (2) for $s = s_1s_2$. □

From Theorem 2, it follows that the set of all generally invertible elements forms a subsemigroup. We will call a semigroup with commuting idempotents a *generalised group* if all its elements are generally invertible. It is easy to see that a generalised group forms an ordinary group if, and only if, it contains a single idempotent. By (4), for generalised groups, the involution $\overline{s} = s^{-1}$ is an inverse automorphism.[3] We define a binary relation $<$ between elements of a generalised group S by the formula

$$(s_1 < s_2) \Leftrightarrow (s_1 s_2^{-1} s_1 = s_1). \tag{5}$$

From this definition it follows that

$$(s_1 < s_2) \Leftrightarrow (s_1^{-1} < s_2^{-1}), \tag{6}$$

hence the inverse automorphism $\overline{s} = s^{-1}$ preserves the binary relation $<$.

Lemma. *The binary relation (5) can also be defined by means of the following formulae:*

$$\begin{aligned}(s_1 < s_2) &\Leftrightarrow (s_1 s_1^{-1} s_2 = s_1) \Leftrightarrow (s_2 s_1^{-1} s_1 = s_1)\\ &\Leftrightarrow (s_1^{-1} s_2 = s_1^{-1} s_1) \Leftrightarrow (s_1 s_2^{-1} = s_1 s_1^{-1}).\end{aligned} \tag{7}$$

Proof. By (1) and (5), $(s_1 < s_2) \Rightarrow (s_2^{-1}s_1 \in I)$, whence $(s_1 < s_2) \Rightarrow (s_2^{-1}s_1 = s_1^{-1}s_2)$, since $s_2^{-1}s_1$ is self-inverse. Then $(s_1 < s_2) \Rightarrow (s_1s_1^{-1}s_2 = s_1)$. Conversely, $(s_1s_1^{-1}s_2 = s_1) \Rightarrow (s_1(s_1s_1^{-1}s_2)^{-1}s_2 = s_1) \Rightarrow (s_1s_2^{-1}s_1 = s_1) \Rightarrow (s_1 < s_2)$. In a similar way, $(s_1 < s_2) \Leftrightarrow (s_1s_1^{-1}s_2 = s_1)$. Now, using (6), we have: $(s_1 < s_2) \Leftrightarrow (s_1^{-1}s_1s_2^{-1} = s_1^{-1}) \Leftrightarrow (s_2s_1^{-1}s_1 = s_1)$. Notice that $(s_1s_1^{-1}s_2 = s_1) \Rightarrow (s_1^{-1}s_1s_1^{-1}s_2 = s_1^{-1}s_1) \Rightarrow (s_1^{-1}s_2 = s_1^{-1}s_1)$, so we get $(s_1 < s_2) \Rightarrow (s_1^{-1}s_2 = s_1^{-1}s_1)$. Conversely, $(s_1^{-1}s_2 = s_1^{-1}s_1) \Rightarrow (s_1s_1^{-1}s_2 = s_1s_1^{-1}s_1 = s_1) \Rightarrow (s_1 < s_2)$. Thus $(s_1 < s_2) \Leftrightarrow (s_1^{-1}s_2 = s_1^{-1}s_1)$. Finally, by (6), $(s_1 < s_2) \Leftrightarrow (s_1s_2^{-1} = s_1s_1^{-1})$.[4] □

Theorem 3. *The binary relation $<$ is a partial order.*

Proof. Reflexivity of $<$ follows immediately from (5) and (3).

By substituting $s_1^{-1}s_1 = s_1^{-1}s_2$ into the equation $s_2s_1^{-1}s_1 = s_1$, we obtain $(s_1 < s_2) \Rightarrow (s_2s_1^{-1}s_2 = s_1)$, by (7). Antisymmetry of $<$ then follows from $(s_2 < s_1) \Rightarrow (s_2s_1^{-1}s_2 = s_2)$, by (5).

It only remains to show transitivity of $<$. Using (7), we obtain $(s_1 < s_2) \wedge (s_2 < s_3) \Rightarrow \{s_1s_3^{-1}s_1 = s_1s_1^{-1}s_2s_3^{-1}s_2s_1^{-1}s_1 = s_1s_1^{-1}s_2s_1^{-1}s_1 = s_1s_1^{-1}s_1 = s_1\}$, i.e., $(s_1 < s_2) \wedge (s_2 < s_3) \Rightarrow (s_1 < s_3)$. □

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[3] *Let S be a semigroup. A bijection $\varphi : S \to S$ is called an* inverse automorphism *(or* anti-automorphism*) if, for all $x, y \in S$, $\varphi(x)\varphi(y) = \varphi(yx)$—see* [5, Section 1.17].

[4] *In the original paper, this last sentence is misprinted as: "Finally, by (6), $(s_1 < s_2) \Leftrightarrow (\boldsymbol{s_2}s_2^{-1} = s_1s_1^{-1})$."*

A binary relation on an arbitrary semigroup S is said to be *closed* with respect to the operation in S if

$$(s_1, s_2) \in \rho \ \wedge \ (s_1^*, s_2^*) \in \rho \Rightarrow (s_1 s_1^*, s_2 s_2^*) \in \rho.$$

It is easy to see that the intersection of two closed binary relations is also a closed binary relation.

Theorem 4. *The partial order $<$ is closed with respect to the operation of a generalised group S.*

Proof. We have $(s_1 < s_2) \wedge (s_1^* < s_2^*) \Rightarrow \{s_1 s_1^* (s_2 s_2^*)^{-1} s_1 s_1^* = s_1 s_1^* s_2^{*-1} s_2^{-1} s_1 s_1^* = s_1 s_2^{-1} s_1 s_1^* s_2^{*-1} s_1^* = s_1 s_1^*\}$, whence

$$(s_1 < s_2) \wedge (s_1^* < s_2^*) \Rightarrow (s_1 s_1^* < s_2 s_2^*). \tag{8}$$

□

Theorem 5. *Every symmetric semigroup of partial one-one transformations of some set forms a generalised group with respect to the operation of multiplication of partial transformations, moreover, its idempotents are the partial identity transformations, its generally invertible elements are the invertible transformations, and the partial order $<$ is restriction of partial transformations.*

Proof. This follows immediately by comparing the theory of generalised groups with the theory of symmetric semigroups of partial one-one transformations [1]. □

Theorem 6. *Every generalised group may be represented as a generalised group of partial one-one transformations.*

In order to prove this theorem, we need the following two lemmas:

Lemma 1. *For every $s \in S$,*

$$Ss = Ss^{-1}s. \tag{9}$$

Proof. $Ss = Sss^{-1}s \subseteq Ss^{-1}s$; conversely, $Ss^{-1} \subseteq Ss$, whence follows (9). □

Lemma 2. *For all $s_1, s_2 \in S$,*

$$Ss_1 \cap Ss_2 = Ss_1 s_2^{-1} s_2 = Ss_2 s_1^{-1} s_1. \tag{10}$$

Proof. Let $i_1, i_2 \in I$. Then $(Si_1 \cap Si_2)i_2 = Si_1 \cap Si_2$.[5] Since $(Si_1 \cap Si_2)i_2 \subseteq Si_1 i_2$, we have $Si_1 \cap Si_2 \subseteq Si_1 i_2$. Conversely, using $Si_1 i_2 = Si_2 i_1$, we obtain $Si_1 i_2 \subseteq Si_1$ and $Si_1 i_2 \subseteq Si_2$, so $Si_1 i_2 \subseteq Si_1 \cap Si_2$. We have shown that $Si_1 \cap Si_2 = Si_1 i_2$. By (9), we have $Ss_1 \cap Ss_2 = Ss_1^{-1} s_1 \cap Ss_2^{-1} s_2 = Ss_1^{-1} s_1 s_2^{-1} s_2 = Ss_1 s_2^{-1} s_2 = Ss_2 s_1^{-1} s_1$. □

[5] *We know that i_2 is a right identity for Si_2. Since $Si_1 \cap Si_2 \subseteq Si_1 i_2$, i_2 is also a right identity for $(Si_1 \cap Si_2)i_2$. My thanks to John Fountain for helping me to see that this is obvious!*

Proof of Theorem 6. Each element $s \in S$ defines a transformation $\overline{s}^* = \overline{s}s$ of the set S to itself, which we call *right translation* by this element. In general, right translation is not by means of one-one transformations. A *reduced right translation* by an element s is a transformation which is only defined in the subset Ss^{-1} and in which it coincides with right translation. By (9), the reduced right translation corresponding to the element s defines a mapping of the subset Ss^{-1} to the subset Ss, in which is defined the reduced right translation corresponding to s^{-1}. Notice that since $(\overline{s} \in Ss^{-1}) \Rightarrow (\overline{s}ss^{-1} = \overline{s})$, the image $\overline{s}s$ of $\overline{s}$ under the reduced right translation corresponding to s is mapped by the reduced right translation corresponding to s^{-1} to the initial element $\overline{s}$. It then follows that a reduced right translation is a partial one-one transformation of S and that the inverse for this translation is the reduced right translation corresponding to the generalised inverse element. In this way, we obtain a representation θ of the generalised group S by the semigroup $\mathfrak{M}(S \times S)$ of all partial one-one transformations of the set S. We show that the representation θ is one-one. Notice that[6] $(\theta(s_1) = \theta(s_2)) \Rightarrow (\forall s \in S)\{(ss_1^{-1}s_1 = ss_1^{-1}s_2) \wedge (ss_2^{-1}s_2 = ss_2^{-1}s_1)\}$. By putting $s = s_1$ and $s = s_2$ in turn, we obtain $(\theta(s_1) = \theta(s_2)) \Rightarrow (s_1s_1^{-1}s_2 = s_1) \wedge (s_2s_2^{-1}s_1 = s_2)$ or, by (7), $(\theta(s_1) = \theta(s_2)) \Rightarrow (s_1 < s_2) \wedge (s_2 < s_1)$, whence $(\theta(s_1) = \theta(s_2)) \Leftrightarrow (s_1 = s_2)$.

In order to complete the proof that θ is an isomorphism from S to the symmetric subsemigroup $\theta(S)$ of the semigroup $\mathfrak{M}(S \times S)$, we must show that θ is indeed a morphism. It is sufficient to show that[7] $\operatorname{dom}(\theta(s_2)\theta(s_1)) = S(s_1s_2)^{-1}$. We know that

$$\operatorname{dom}(\theta(s_2)\theta(s_1)) = (\theta(s_1))^{-1}\{\operatorname{im}\theta(s_1) \cap \operatorname{dom}\theta(s_2)\},$$

so, by (10),

$$\operatorname{dom}(\theta(s_2)\theta(s_1)) = (Ss_1 \cap Ss_2^{-1})s_1^{-1} = Ss_2^{-1}s_1^{-1}s_1s_1^{-1} = Ss_2^{-1}s_1^{-1} = S(s_1s_2)^{-1}.$$

□

Amongst the different classes of generalised groups, those of particular interest are the generalised groups with zero in which the product of any two distinct idempotents is zero, since these coincide with the partial groups of Croisot [2], with zero adjoined. The identity elements of these partial groups are the idempotents of the generalised group, other than zero, and the invertible elements are the generally invertible elements of the generalised group. The partial order in the generalised group is the identity relation in the partial group. As is well-known, a special case of the generalised groups of Croisot are the groupoids of Brandt [3]. In this way, the theory of Brandt groupoids leads into the theory of generalised groups.

[6] *Wagner wrote simply "(s)", rather than "$(\forall s \in S)$".*

[7] *Wagner denoted* dom *by* pr_1 *(first projection) and* im *by* pr_2 *(second projection)—see* [6].

Some notes on the translation

First of all, the footnotes in italics are mine, rather than Wagner's; similarly, the starred references. I have tried to stick to a literal translation wherever possible, though I have made a number of simplifications along the way. For example, I use the word *idempotent* as a noun, whereas Wagner used it only as an adjective, referring throughout to *idempotent elements* (*идемпотентные елементы*).

A note on the author's name: Viktor Vladimirovich Wagner (Виктор Владимирович Вагнер). It would perhaps make more sense to transliterate the *B* of *Вагнер* as a *V*, as I have done with both *Виктор* and *Владимирович* (this is the AMS convention). I have spelt *Wagner* with a *W* as this seems to have been Wagner's own preferred form, since his father was German. According to Boris Schein [6], Wagner was amused by the alternative spelling of his name.

References

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